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# DYNAMIC EQUATIONS FOR A TWO-LINK FLEXIBLE ROBOT ARM

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**Abstract**—The dynamic equations for a two-link flexible robot arm have been derived rigorously. The arm is moving in the vertical plane. The payload is simulated by attaching additional masses to the arm at any specific locations. Although the governing equations of the system and the measurements are nonlinear, they are explicitly obtained. The control strategy and the general procedures to construct a linear observer and to formulate a control law are discussed.

## 1. INTRODUCTION

Most of today's analyses and controls of industrial robots are based on the assumption that the robot arm is just a collection of rigid bodies so that, after the joint angles are driven to assume the pre-computed values, the end effector of the robot arm, by dead reckoning, will be in the intended position. Therefore, most of the robots are built to be massive and unwieldy.

Clearly, it is desirable to build lightweight robot arms which have a large working volume, high mobility and the capability to carry heavy payloads. In order to meet these requirements, the robot arm has to be flexible, in other words, the rigid-body-assumption in robotics has to be abandoned. Then the deflection and the vibration of the robot arm present a severe problem to the accuracy and the stability in positioning. Therefore the control of flexible manipulators is becoming a critical issue in robotics.

Cannon and Schmitz [1] published the pioneer work in the area of control of flexible robot arms, in 1984. In that work the mathematical modeling and the initial experiments have been carried out to address the control of a one-link flexible robot arm where the position of the end effector (tip) is controlled by measuring that position and using the measurement as a basis for applying control torque to the other end of the arm (joint). Also, it is worthwhile to mention the works of Harashima and Ueshiba [2], Wang and Vidyasagar [3, 4], Sangveraphunsiri [5], and Book *et al.* [6]. In all those works there are two things in common: the one-link robot arm, with its hub rotating about the $z$-axis, sweeps the horizontal $x$–$y$ plane; the flexible arm is modeled as a beam whose deflection is represented by a series in terms of eigenfunctions (normal modes). Lee *et al.* [7], Lee and Wang [8] rigorously derived the dynamic equations and designed the control system for a one-link arm which has two degrees of freedom in rotation and one in translation so that the working volume of the end effector is a three-dimensional space instead of a cycle on the horizontal plane. Usoro *et al.* [9] presented a finite element/Langrangian approach for the mathematical modeling of a two-link flexible manipulator.

In this work the dynamic equations for a two-link flexible robot arm, moving in the vertical ($x$–$z$) plane are rigorously derived. The payload is simulated by attaching additional masses to the arm at any specified locations. Finite element method, based on elementary beam theory, has been employed during the process of formulation. The explicit form of the nonlinear governing equations for the mechanical system has been obtained. It is assumed that the position of the end effector can be measured and only the information of that measurement will be used as a feedback to the control system. The general procedures to construct a linear observer and to formulate a control law are discussed. However, how to find the control gain matrix and the estimate gain matrix is left for future study.

## 2. PROBLEM DESCRIPTION

The undeformed configuration of a two-link robot arm is shown in Fig. 1. In this work it is considered that the motion of the arm is confined in the vertical plane, i.e., the $x$–$z$ plane. It is seen that in Fig. 1, the upper arm makes an angle $\phi$ with respect to the $z$-axis, which is opposite to the direction of gravity, and the angle between the upper arm and the lower arm is denoted by $\gamma$. The original lengths of the upper arm and the lower arm are denoted by $l^1$ and $l^2$, respectively. The deformed configuration of the two-link robot arm is shown in Fig. 2. Define two new coordinate systems, $(x^1, z^1)$ and $(x^2, z^2)$ as shown in Fig. 2, such that the $x^1$-axis and the $x^2$-axis are parallel to the tangents of the upper arm and the lower arm at the origin and at the joint between the two links, respectively. Let the angle between the $x^2$-axis and the $z$-axis be denoted by $\beta$. Model the upper arm and the lower arm by $n$ beam elements and $m$ beam elements, respectively. Then there are

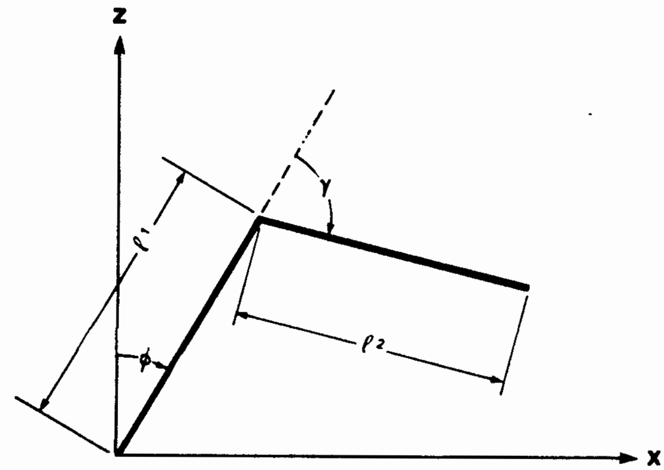


Fig. 1. The undeformed configuration of a two-link robot arm.

$n + m + 1$ nodal points and each is associated with a lumped mass. The payloads may also be simulated by the masses attached to some nodal points. From now on, unless otherwise stated, the lumped mass at any nodal point stands for the sum of the payload carried at the point and the mass of the beam distributed to that point. The position vector of the generic $i$th nodal point ($i = 0, 1, 2, \ldots, n$) of the deformed upper arm can be expressed in the $(x^1, z^1)$ coordinate system as

$$\mathbf{x}_i^1 \equiv (x_i^1, z_i^1) = (X_i^1, U_i^1), \tag{1}$$

where $U_i^1$ is the displacement of the $i$th nodal point in the direction of $z^1$-axis and the lumped mass at this point is denoted by $M_i^1$; Similarly, the $j$th nodal point ($j = 1, 2, \ldots, m$) of the deformed lower arm occupies

$$\mathbf{x}_j^2 \equiv (x_j^2, z_j^2) = (X_j^2, U_j^2); \tag{2}$$

and the lumped mass at this point is denoted by $M_j^2$.

### 3. TRANSFORMATIONS

The position vector for any point on the upper arm, expressed in the global coordinate system $(\mathbf{x}, \mathbf{z})$, may be obtained as

$$\mathbf{x} = \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \sin\phi & -\cos\phi \\ \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} X^1 \\ U^1 \end{bmatrix}$$

$$\equiv \mathbf{Q}^1 \begin{bmatrix} X^1 \\ U^1 \end{bmatrix} \equiv \mathbf{Q}^1 \mathbf{x}^1. \tag{3}$$

It is noticed that $\mathbf{Q}^1$ is an orthogonal transformation matrix which has the following property

$$(\mathbf{Q}^1)^{-1} = (\mathbf{Q}^1)^T. \tag{4}$$

In other words, any vector $\mathbf{V}$, in the global coordinate system, can be transformed into $\mathbf{V}^1$, in the $(x^1, z^1)$ coordinate system, through $\mathbf{V}^1 = (\mathbf{Q}^1)^T \mathbf{V}$.

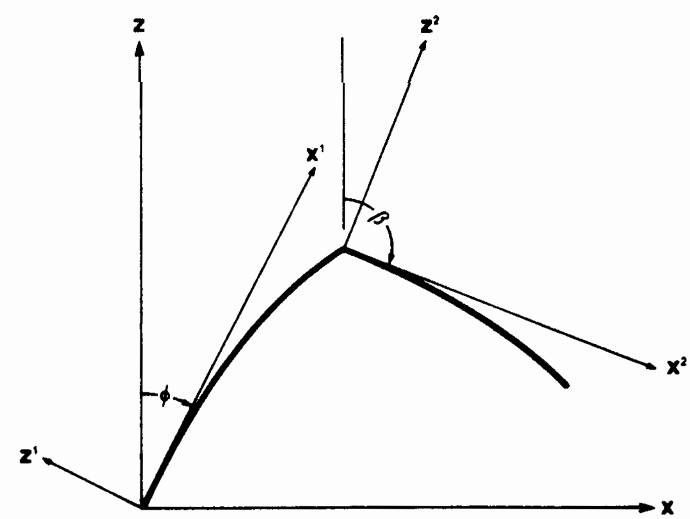


Fig. 2. The deformed configuration of the two-link robot arm and the coordinate systems.

Now the velocity and the acceleration can be obtained as

$$\mathbf{v} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{\phi}\cos\phi & \dot{\phi}\sin\phi \\ -\dot{\phi}\sin\phi & \dot{\phi}\cos\phi \end{bmatrix} \begin{bmatrix} X^1 \\ U^1 \end{bmatrix} + \mathbf{Q}^1 \begin{bmatrix} 0 \\ \dot{U}^1 \end{bmatrix}$$

$$\equiv \dot{\mathbf{Q}}^1 \mathbf{x}^1 + \mathbf{Q}^1 \mathbf{v}^1, \tag{5}$$

$$\mathbf{a} \equiv \begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{\phi}\cos\phi - \dot{\phi}\dot{\phi}\sin\phi & \ddot{\phi}\sin\phi + \dot{\phi}\dot{\phi}\cos\phi \\ -\ddot{\phi}\sin\phi - \dot{\phi}\dot{\phi}\cos\phi & \ddot{\phi}\cos\phi - \dot{\phi}\dot{\phi}\sin\phi \end{bmatrix}$$

$$\times \begin{bmatrix} X^1 \\ U^1 \end{bmatrix} + 2\dot{\mathbf{Q}}^1 \mathbf{v}^1 + \mathbf{Q}^1 \begin{bmatrix} 0 \\ \ddot{U}^1 \end{bmatrix}$$

$$\equiv \ddot{\mathbf{Q}}^1 \mathbf{x}^1 + 2\dot{\mathbf{Q}}^1 \mathbf{v}^1 + \mathbf{Q}^1 \mathbf{a}^1. \tag{6}$$

The position vector of any point on the lower arm, expressed in the global coordinate system $(\mathbf{x}, \mathbf{z})$, may be obtained as

$$\mathbf{x} \equiv \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} X^2 \\ U^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \sin\phi & -\cos\phi \\ \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} l^1 \\ U^* \end{bmatrix}$$

$$\equiv \mathbf{Q}^2 \mathbf{x}^2 + \mathbf{Q}^1 \mathbf{x}_n^1, \tag{7}$$

where

$$U^* \equiv U_n^1. \tag{8}$$

Then the velocity and the acceleration can be obtained as

$$\mathbf{v} = \dot{\mathbf{Q}}^2 \mathbf{x}^2 + \mathbf{Q}^2 \mathbf{v}^2 + \dot{\mathbf{Q}}^1 \mathbf{x}_n^1 + \mathbf{Q}^1 \mathbf{v}_n^1, \tag{9}$$

$$\mathbf{a} = \ddot{\mathbf{Q}}^2 \mathbf{x}^2 + 2\dot{\mathbf{Q}}^2 \mathbf{v}^2 + \mathbf{Q}^2 \mathbf{a}^2 + \ddot{\mathbf{Q}}^1 \mathbf{x}_n^1 + 2\dot{\mathbf{Q}}^1 \mathbf{v}_n^1 + \mathbf{Q}^1 \mathbf{a}_n^1, \tag{10}$$

where

$$\mathbf{v}^2 \equiv \begin{bmatrix} 0 \\ \dot{U}^2 \end{bmatrix}, \quad \mathbf{a}^2 \equiv \begin{bmatrix} 0 \\ \ddot{U}^2 \end{bmatrix}, \tag{11}$$

$$\mathbf{v}_n^1 \equiv \begin{bmatrix} 0 \\ \dot{U}^* \end{bmatrix}, \quad \mathbf{a}_n^1 \equiv \begin{bmatrix} 0 \\ \ddot{U}^* \end{bmatrix}. \tag{12}$$

## 4. INERTIA FORCE AND GRAVITY

The total force acting on a generic point is equal to the sum of the inertia force and the gravitational force acting on the point, i.e.,

$$\mathbf{f} = -M\mathbf{a} - Mg \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{13}$$

The total force acting on the $i$th nodal point of the upper arm, expressed in the $(x^1, z^1)$ coordinate system, can be calculated as

$$\mathbf{f}_i^1 = (\mathbf{Q}^1)^T \mathbf{f}_i$$

$$= -M_i^1 (\mathbf{Q}^1)^T \left[ \ddot{\mathbf{Q}}^1 \mathbf{x}^1 + 2\dot{\mathbf{Q}}^1 \mathbf{v}^1 + \mathbf{Q}^1 \mathbf{a}^1 + \begin{bmatrix} 0 \\ g \end{bmatrix} \right]. \tag{14}$$

Explicitly, eqn. (14) can be rewritten as

$$f_i^1(x) = -M_i^1 [U_i^1 \ddot{\phi} - X_i^1 \dot{\phi}\dot{\phi} + 2\dot{U}_i^1 \dot{\phi} + g\cos\phi], \tag{14.1}$$

$$f_i^1(z) = -M_i^1 [\ddot{U}_i^1 - X_i^1 \ddot{\phi} - U_i^1 \dot{\phi}\dot{\phi} + g\sin\phi]. \tag{14.2}$$

The total force acting on the $j$th nodal point of the lower arm can be expressed in the $(x^1, z^1)$ coordinate system and the $(x^2, z^2)$ coordinate system, respectively, as follows

$$\mathbf{f}_j^{21} = -M_j^2 (\mathbf{Q}^1)^T \left[ \ddot{\mathbf{Q}}^2 \mathbf{x}^2 + 2\dot{\mathbf{Q}}^2 \mathbf{v}^2 + \mathbf{Q}^2 \mathbf{a}^2 + \ddot{\mathbf{Q}}^1 \mathbf{x}_n^1 + 2\dot{\mathbf{Q}}^1 \mathbf{v}_n^1 + \mathbf{Q}^1 \mathbf{a}_n^1 + \begin{bmatrix} 0 \\ g \end{bmatrix} \right], \tag{15}$$

$$\mathbf{f}_j^{22} = -M_j^2 (\mathbf{Q}^2)^T \left[ \ddot{\mathbf{Q}}^2 \mathbf{x}^2 + 2\dot{\mathbf{Q}}^2 \mathbf{v}^2 + \mathbf{Q}^2 \mathbf{a}^2 + \ddot{\mathbf{Q}}^1 \mathbf{x}_n^1 + 2\dot{\mathbf{Q}}^1 \mathbf{v}_n^1 + \mathbf{Q}^1 \mathbf{a}_n^1 + \begin{bmatrix} 0 \\ g \end{bmatrix} \right]. \tag{16}$$

Explicitly, eqns (15–16) can be rewritten as

$$f_j^{21}(x) = -M_j^2 [U^* \ddot{\phi} - l^1 \dot{\phi}\dot{\phi} + 2\dot{U}^* \dot{\phi} + g\cos\phi + S\ddot{U}_j^2 + 2C\dot{\beta}\dot{U}_j^2 + (-SX_j^2 + CU_j^2)\ddot{\beta} - (CX_j^2 + SU_j^2)\dot{\beta}\dot{\beta}], \tag{15.1}$$

$$f_j^{21}(z) = -M_j^2 [\ddot{U}^* - l^1 \ddot{\phi} - U^* \dot{\phi}\dot{\phi} + g\sin\phi + C\ddot{U}_j^2 - 2S\dot{\beta}\dot{U}_j^2 - (CX_j^2 + SU_j^2)\ddot{\beta} + (SX_j^2 - CU_j^2)\dot{\beta}\dot{\beta}], \tag{15.2}$$

$$f_j^{22}(x) = -M_j^2 [U_j^2 \ddot{\beta} - X_j^2 \dot{\beta}\dot{\beta} + 2\dot{U}_j^2 \dot{\beta} + g\cos\beta - S\ddot{U}^* + 2C\dot{U}^* \dot{\phi} + (Sl^1 + CU^*)\ddot{\phi} - (Cl^1 - SU^*)\dot{\phi}\dot{\phi}], \tag{16.1}$$

$$f_j^{22}(z) = -M_j^2 [\ddot{U}_j^2 - X_j^2 \ddot{\beta} - U_j^2 \dot{\beta}\dot{\beta} + g\sin\beta + C\ddot{U} + 2S\dot{U}^* \dot{\phi} - (Cl^1 - SU^*)\ddot{\phi} - (Sl^1 + CU^*)\dot{\phi}\dot{\phi}], \tag{16.2}$$

where $C \equiv \cos(\beta - \phi)$ and $S \equiv \sin(\beta - \phi)$.

The total force acting on the lower arm is equivalent to a force in the $x^1$-direction, $F_x$, a force in the

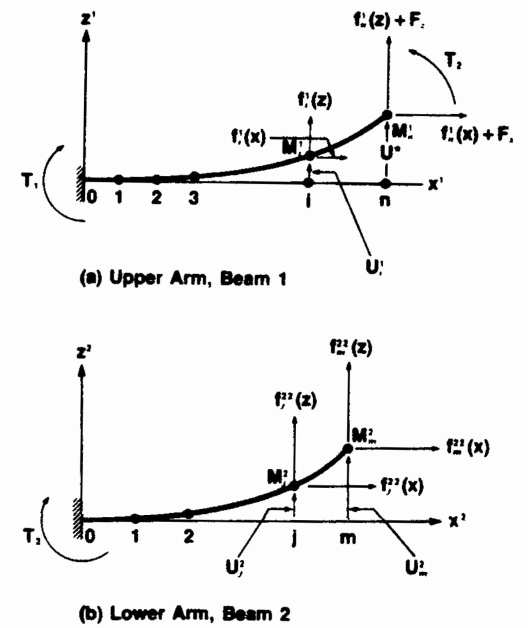


Fig. 3. The forces and moment acting on the two cantilever beams.

$z^1$-direction, $F_z$, and a bending moment, $T_2$, acting at the joint between the upper arm and the lower arm (cf. Fig. 3). These resultant forces and moment may be written as

$$
\begin{aligned}
F_x &= \sum_{j=1}^{m} f_j^{21}(x) \\
&= -\Gamma(U^*\ddot{\phi} - l^1\dot{\phi}\dot{\phi} + 2\dot{U}^*\dot{\phi} + g\cos\phi) \\
&\quad - \sum_{j=1}^{m} M_j^2[S\ddot{U}_j^2 + (-SX_j^2 + CU_j^2)\ddot{\beta} + 2C\dot{\beta}\dot{U}_j^2 \\
&\quad - (CX_j^2 + SU_j^2)\dot{\beta}\dot{\beta}],
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
F_z &= \sum_{j=1}^{m} f_j^{21}(z) \\
&= -\Gamma(\ddot{U}^* - l^1\ddot{\phi} - U^*\dot{\phi}\dot{\phi} + g\sin\phi) \\
&\quad - \sum_{j=1}^{m} M_j^2[C\ddot{U}_j^2 - (CX_j^2 + SU_j^2)\ddot{\beta} - 2S\dot{\beta}\dot{U}_j^2 \\
&\quad + (SX_j^2 - CU_j^2)\dot{\beta}\dot{\beta}],
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
T_2 &= \sum_{j=1}^{m} [f_j^{22}(z)X_j^2 - f_j^{22}(x)U_j^2] \\
&= \sum_{j=1}^{m} M_j^2[X_j^2\ddot{U}_j^2 + (CX_j^2 + SU_j^2)\ddot{U}^* \\
&\quad - (Cl^1X_j^2 - SU^*X_j^2 + Sl^1U_j^2 + CU^*U_j^2)\ddot{\phi} \\
&\quad - (X_j^2X_j^2 + U_j^2U_j^2)\ddot{\beta} + 2\dot{U}^*(SX_j^2 - CU_j^2)\dot{\phi} \\
&\quad - 2U_j^2\dot{U}_j^2\dot{\beta} + (\sin\beta X_j^2 - \cos\beta U_j^2)g \\
&\quad - (Sl^1X_j^2 + CU^*X_j^2 - Cl^1U_j^2 + SU^*U_j^2)\dot{\phi}\dot{\phi}],
\end{aligned} \tag{19}
$$

where

$$\Gamma \equiv \sum_{j=1}^{m} M_j^2.$$

Similarly, the total moment acting on the origin, $T_1$, can be obtained as

$$
\begin{aligned}
T_1 &= \sum_{i=1}^{n} [f_i^1(z)X_i^1 - f_i^1(x)U_i^1] + F_z l^1 - F_x U^* + T_2 \\
&= -\sum_{i=1}^{n-1} M_i^1 X_i^1 \ddot{U}_i^1 - (M_n^1 + \Gamma)l^1\ddot{U}^* \\
&\quad - (Cl^1 - SU^*)\sum_{j=1}^{m} M_j^2\ddot{U}_j^2 \\
&\quad + \ddot{\phi}\left[\Gamma(l^1l^1 + U^*U^*) + \sum_{i=1}^{n} M_i^1(X_i^1X_i^1 + U_i^1U_i^1)\right] \\
&\quad + \ddot{\beta}\left\{\sum_{j=1}^{m} M_j^2[(Cl^1 - SU^*)X_j^2 + (Sl^1 + C\dot{U})U_j^2]\right\} \\
&\quad - g\left[\Gamma(l^1\sin\phi - U^*\cos\phi)\right. \\
&\quad \left. + \sum_{i=1}^{n} M_i^1(\sin\phi X_i^1 - \cos\phi U_i^1)\right] \\
&\quad + 2\dot{\phi}\left(\Gamma U^*\dot{U}^* + \sum_{i=1}^{n} M_i^1 U_i^1\dot{U}_i^1\right) \\
&\quad + 2\dot{\beta}(Sl^1 + CU^*)\sum_{j=1}^{m} M_j^2\dot{U}_j^2 \\
&\quad + \dot{\beta}\dot{\beta}\left\{\sum_{j=1}^{m} M_j^2[(Cl^1 - SU^*)U_j^2\right. \\
&\quad \left. - (Sl^1 + CU^*)X_j^2]\right\} + T_2.
\end{aligned} \tag{20}
$$

Now, the two-link robot arm can be treated as two cantilever beams on which the forces and moment, due to inertia force and gravity, are acting, as shown in Fig. 3.

## 5. FINITE ELEMENT ANALYSIS

Following standard procedures in finite element analysis [10, 11, 12] and the elementary beam theory, one may obtain the governing equations for the lower arm (beam 2), which is treated as a cantilever beam, as follows [7]

$$\mathbf{K}^2\mathbf{U}^2 = \mathbf{f}^{22}, \tag{21}$$

where

$$\mathbf{U}^2 \equiv (U_1^2, U_2^2, U_3^2, \ldots, U_m^2)^T, \tag{22}$$

$$\mathbf{f}^{22} \equiv [f_1^{22}(z), f_2^{22}(z), f_3^{22}(z), \ldots, f_m^{22}(z)]^T, \tag{23}$$

and $\mathbf{K}^2$ is the $(m \times m)$ stiffness matrix for a cantilever beam subjected to applied forces only. For the upper arm (beam 1), because there is a bending moment, $T_2$, acting at the free end of beam 1 (cf. Fig. 3), following the same procedure outlined in [7], the governing equations may be written as

$$\begin{bmatrix} \mathbf{K}' & \mathbf{K} \\ \mathbf{K}^T & k \end{bmatrix} \begin{bmatrix} \mathbf{U}^1 \\ s \end{bmatrix} = \begin{bmatrix} \mathbf{f}' \\ T_2 \end{bmatrix}, \tag{24}$$

where

$$\mathbf{U}^1 \equiv (U_1^1, U_2^1, U_3^1, \ldots, U_n^1)^T, \tag{25}$$

$$\mathbf{f}' = (f^1(z), f_2^1(z), f_3^1(z), \ldots, f_n^1(z) + F_z), \tag{26}$$

and $s$ is the slope at the free end of beam 1; $\mathbf{K}'$ is a $(n \times n)$ matrix; $\mathbf{K}$ is a vector of length $n$ and it can be written as $(K_1, K_2, K_3, \ldots, K_n)^T$. By eliminating $s$ from eqn (24), the following is obtained

$$\mathbf{K}^1\mathbf{U}^1 = \mathbf{f}' - \mathbf{K}T_2/k, \tag{27}$$

where

$$\mathbf{K}^1 = \mathbf{K}' - \mathbf{K}\mathbf{K}^T/k. \tag{28}$$

Now eqns (21, 27, 19, 20) may be rewritten in a more compact form as follows

$$\begin{bmatrix} \mathbf{a}_1 & 0 & \mathbf{b} \\ 0 & \mathbf{a}_2 & \mathbf{c} \\ \mathbf{b}^T & \mathbf{c}^T & \mathbf{a}_3 \end{bmatrix} \ddot{\omega} \equiv \mathbf{A}^* \ddot{\omega} = \mathbf{\Omega}, \tag{29}$$

where

$$\omega \equiv (U_1^1, U_2^1, \ldots, U_{n-1}^1, U_1^2, U_2^2, \ldots U_m^2, U_n^1, \beta, \phi,)^T, \tag{30}$$

$$\mathbf{a}_1 \equiv \mathrm{diag}[M_1^1, M_2^1, M_3^1, \ldots, M_{n-1}^1], \tag{31}$$

$$\mathbf{a}_2 \equiv \mathrm{diag}[M_1^2, M_2^2, M_3^2, \ldots, M_m^2], \tag{32}$$

$$\mathbf{a}_3 \equiv \begin{bmatrix} A_1 & A_4 & -l^1A_1 \\ A_4 & A_2 & A_5 \\ -l^1A_1 & A_5 & A_3 \end{bmatrix}, \tag{33}$$

$$A_2 \equiv \sum_{j=1}^{m} M_j^2[X_j^2X_j^2 + U_j^2U_j^2], \tag{37.2}$$

$$A_3 \equiv \sum_{i=1}^{n} M_i^1[X_i^1X_i^1 + U_i^1U_i^1] + \Gamma[l^1l^1 + U_n^1U_n^1], \tag{37.3}$$

$$A_4 \equiv -C\sum_{j=1}^{m} M_j^2X_j^2 - S\sum_{j=1}^{m} M_j^2U_j^2, \tag{37.4}$$

$$A_5 \equiv (l^1C - U^*S)\sum_{j=1}^{m} M_j^2X_j^2 + (l^1S + U^*C)\sum_{j=1}^{m} M_j^2U_j^2, \tag{37.5}$$

$$c_{1j} \equiv CM_j^2, \tag{38.1}$$

$$c_{2j} \equiv -M_j^2X_j^2, \tag{38.2}$$

$$c_{3j} \equiv M_j^2(U^*S - l^1C), \tag{38.3}$$

$$p_i = \sum_{j=1}^{n} K_{ij}^1U_j^1 - M_i^1 g\sin\phi + M_i^1U_i^1\dot{\phi}\dot{\phi} - (K_i/k)T_2, \tag{39}$$

$$q_i = -\sum_{j=1}^{m} K_{ij}^2U_j^2 - M_j^2 g\sin\beta - 2S\dot{U}^*\dot{\phi}M_i^2 + \dot{\beta}\dot{\beta}U_i^2M_i^2 + (Sl^1 + CU^*)\dot{\phi}\dot{\phi}M_i^2, \tag{40}$$

$$\gamma_1 = -\sum_{j=1}^{n} K_{nj}^1U_j^1 + (M_n^1 + \Gamma)U^*\dot{\phi}\dot{\phi} - (K_n/k)T_2 - (M_n^1 + \Gamma)g\sin\phi + 2S\dot{\beta}\sum_{j=1}^{m} M_j^2\dot{U}_j^2 - \dot{\beta}\dot{\beta}\sum_{j=1}^{m}(SX_j^2 - CU_j^2)M_j^2, \tag{41.1}$$

$$\gamma_2 = T_2 + g\sum_{j=1}^{m} M_j^2(\sin\beta X_j^2 - \cos\beta U_j^2) + 2\dot{U}^*\dot{\phi}\sum_{j=1}^{m} M_j^2(SX_j^2 - CU_j^2)$$

$$\mathbf{b}^T \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -M_1^1X_1^1 & -M_2^1X_2^1 & -M_3^1X_3^1 & \cdots & -M_{n-1}^1X_{n-1}^1 \end{bmatrix}, \tag{34}$$

$$\mathbf{c}^T \equiv \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1m} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2m} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3m} \end{bmatrix}, \tag{35}$$

$$\mathbf{\Omega} \equiv (p_1, p_2, \ldots, p_{n-1}, q_1, q_2, \ldots, q_m, \gamma_1, \gamma_2, \gamma_3), \tag{36}$$

$$A_1 \equiv M_n^1 + \Gamma, \tag{37.1}$$

$$-2\dot{\beta}\sum_{j=1}^{m} M_j^2U_j^2\dot{U}_j^2 - \dot{\phi}\dot{\phi}\sum_{j=1}^{m} M_j^2(Sl^1X_j^2 + CU^*X_j^2 - Cl^1U_j^2 + SU^*U_j^2), \tag{41.2}$$

$$\gamma_3 = T_1 - T_2 + g\left[\Gamma(l^1 \sin\phi - U^* \cos\phi) + \sum_{i=1}^{n} M_i^1(\sin\phi X_i^1 - \cos\phi U_i^1)\right] - 2\dot\phi\left(\Gamma U^* \dot U^* + \sum_{i=1}^{n} M_i^1 U_i^1 \dot U_i^1\right) - 2\dot\beta(Sl^1 + CU^*)\sum_{j=1}^{m} M_j^2 \dot U_j^2 - \dot\beta\dot\beta\left\{\sum_{j=1}^{m} M_j^2[(Cl^1 - SU^*)U_j^2 - (Sl^1 + CU^*)X_j^2]\right\}. \tag{41.3}$$

## 6. THE INVERSE OF $A^*$

In order to proceed with the derivation, it is necessary to write eqn (29) in the following form

$$\ddot\omega = \mathbf{A}\boldsymbol{\Omega}, \tag{42}$$

in other words, it is necessary to find the inverse of $A^*$. However, it is seen that $A^*$ is a function of displacements, $U_1^1, U_2^1, \ldots, U_n^1, U_1^2, U_2^2, \ldots, U_m^2$, and joint angles, $\beta$ and $\phi$, which are time-dependent. Certainly, it is much more desirable if one is able to invert $A^*$ analytically rather than numerically. Fortunately, the $(n+m+2)\times(n+m+2)$ matrix $\mathbf{A}$ can be calculated through the following steps. First, define several variables as

$$a_{11} \equiv M_n^1 + SS\Gamma \tag{43.1}$$

$$a_{12} \equiv -S\sum_{j=1}^{m} M_j^2 U_j^2, \tag{43.2}$$

$$a_{13} \equiv -M_n^1 l^1 - S\Gamma(l^1 S + U^* C), \tag{43.3}$$

$$a_{22} \equiv \sum_{j=1}^{m} M_j^2 U_j^2 U_j^2, \tag{43.4}$$

$$a_{23} \equiv (l^1 S + U^* C)\sum_{j=1}^{m} M_j^2 U_j^2, \tag{43.5}$$

$$a_{33} \equiv (l^1 S + U^* C)(l^1 S + U^* C)\Gamma + M_n^1 l^1 l^1 + \sum_{i=1}^{n} M_i^1 U_i^1 U_i^1, \tag{43.6}$$

$$a'_{22} \equiv a_{22} - a_{12}a_{12}/a_{11} \tag{44.1}$$

$$a'_{23} \equiv a_{23} - a_{13}a_{12}/a_{11} \tag{44.2}$$

$$a'_{33} \equiv a_{33} - a_{13}a_{13}/a_{11} - a'_{23}a'_{23}/a'_{22}, \tag{44.3}$$

$$b_1 \equiv -a_{12}/a_{11}, \tag{45.1}$$

$$b_2 \equiv -a_{13}/a_{11} - b_1 a'_{23}/a'_{22} \tag{45.2}$$

$$b_3 \equiv -a'_{23}/a'_{22}. \tag{45.3}$$

Second, define a vector $\mathbf{V}_k$ $(k = 1, 2, 3, \ldots, n+m+2)$, of length $(n+m+2)$, as follows: $\mathbf{V}_k = 0$ except that

(1) for $k = n+m+2$

$$V_k(n+m+2) = 1.0, \tag{46}$$

(2) for $k = n+m+1$

$$V_k(n+m+1) = 1.0, \quad V_k(n+m+2) = b_3, \tag{47}$$

(3) for $k = n+m$

$$V_k(n+m) = 1.0,\ V_k(n+m+1) = b_1,\ V_k(n+m+2) = b_2, \tag{48}$$

(4) for $k \le n-1$

$$V_k(k) = 1.0, \quad V_k(n+m+2) = X_k^1, \tag{49}$$

(5) for $n \le k < n+m$

$$V_k(k) = 1.0, \quad V_k(n+m) = -C,\quad V_k(n+m+1) = X_{k-n+1}^2 - b_1 C,\quad V_k(n+m+2) = l^1 C - U^* S + b_3 X_{k-n+1}^2 - b_2 C. \tag{50}$$

Now, the $k$th column of matrix $\mathbf{A}$ can be obtained as

$$A(n+m+2, k) = V_k(n+m+2)/a'_{33} \equiv e_1,$$

$$A(n+m+1, k) = [V_k(n+m+1) - a'_{23}e_1]/a'_{22} \equiv e_2,$$

$$A(n+m, k) = [V_k(n+m) - a_{13}e_1 - a_{12}e_2]/a_{11} \equiv e_3,$$

$$A(l, k) = V_k(l)/M_{l-n+1}^2 - Ce_3 + X_{l-n+1}^2 e_2 - (U^* S - l^1 C)e_1, \quad n \le l < n+m$$

$$A(l, k) = V_k(l)/M_l^1 + X_l^1 e_1, \quad l \le n-1. \tag{51}$$

Therefore, one may write symbolically that

$$\mathbf{A} = \mathbf{A}[\omega] \tag{52}$$

to indicate the matrix $\mathbf{A}$ may be expressed explicitly as a function of $\omega$ [cf. eqn (30)].

## 7. TARGET

Consider the displacements and the joint angles, $\omega$, as the state variables and the torques, $T_1$ and $T_2$, as control variables of the system. The purpose of the control is to find the control law that makes the system converge to a steady state which meets certain prescribed requirements. If the solutions are converging, then, as time approaches infinity, the time derivatives of all the variables approach zero, i.e.,

$$\lim_{t\to\infty} \omega = \lim_{t\to\infty} [U_1^1(t), U_2^1(t), \ldots, U_1^2(t), U_2^2(t), \ldots, \beta(t), \phi(t)] = [U_1^{1f}, U_2^{1f}, \ldots, U_1^{2f}, U_2^{2f}, \ldots, \beta^f, \phi^f] \equiv \omega^f. \quad (53)$$

In other words,

$$\mathbf{\Omega}[\omega^f] = 0, \quad (54)$$

which, according to eqns (36, 39, 40, 41), implies

$$\sum_{j=1}^{n} K_{ij}^1 U_j^{1f} = -(K_i/k)T_2^f - M_i^* g \sin\phi^f, \quad (i = 1, 2, \ldots, n) \quad (55)$$

$$\sum_{j=1}^{m} K_{ij}^2 U_j^{2f} = M_i^2 g \sin\beta^f, \quad (i = 1, 2, \ldots, m) \quad (56)$$

$$T_2^f = g \sum_{j=1}^{m} M_j^2 [U_j^{2f}\cos\beta^f - X_j^2 \sin\beta^f], \quad (57)$$

$$T_1^f = T_2^f - g\left[\Gamma(l^1 \sin\phi^f - U_n^{1f}\cos\phi^f) + \sum_{i=1}^{m} M_i^1 (X_i^1 \sin\phi^f - U_i^{1f}\cos\phi^f)\right], \quad (58)$$

where $M_i^* = M_i^1$ for $i = 1, 2, \ldots, n-1$ and $M_n^* = M_n^1 + \Gamma$. The iterative procedure to solve for $\omega^f$ may be described as follows. First, give trial values for $\beta^f$ *and* $\phi^f$. Based on eqn (56), solve for $U_i^{2f}$ $(i = 1, 2, \ldots, m)$ and then $T_2^f$ is calculated according to eqn (57). Now $U_i^{1f}$ $(i = 1, 2, \ldots, n)$ can be obtained by solving eqn (55). Finally, $T_1^f$ can be calculated from eqn (58). In order for the end effector to reach the given target position $(x^t, z^t)$, eqn (7) requires that

$$x^t = l^2 \sin\beta^f - U_m^{2f}\cos\beta^f + l^1 \sin\phi^f - U_n^{1f}\cos\phi^f, \quad (59)$$

$$z^t = l^2 \cos\beta^f + U_m^{2f}\sin\beta^f + l^1 \cos\phi^f + U_n^{1f}\sin\phi^f, \quad (60)$$

which means $\omega^f$, $T_1^f$, and $T_2^f$ may be determined by eqns (55–60).

Define the incremental state variables and the incremental control variables as

$$\Delta\omega \equiv (U_1^1 - U_1^{1f}, U_2^1 - U_2^{1f}, \ldots, U_1^2 - U_1^{2f}, U_2^2 - U_2^{2f}, \ldots, \beta - \beta^f, \phi - \phi^f) \equiv (\Delta U_1^1, \Delta U_2^1, \ldots, \Delta U_1^2, \Delta U_2^2, \ldots, \Delta\beta, \Delta\phi), \quad (61.1)$$

$$\Delta\mathbf{T} \equiv (\Delta T_1, \Delta T_2)^T \equiv (T_1 - T_1^f, T_2 - T_2^f)^T. \quad (61.2)$$

Then eqns (39–41) can be rewritten as

$$p_i = -\sum_{j=1}^{n} K_{ij}^1 \Delta U_j^1 - (K_i/k)\Delta T_2 - M_i^1 g \cos\phi^f \Delta\phi + p_i^N \equiv p_i^L + p_i^N, \quad (i = 1, 2, \ldots, n-1) \quad (39^*)$$

$$q_i = -\sum_{j=1}^{m} K_{ij}^2 \Delta U_j^2 - M_i^2 g \cos\beta^f \Delta\beta + q_i^N \equiv q_i^L + q_i^N, \quad (i = 1, 2, \ldots, m) \quad (40^*)$$

$$\gamma_1 = -\sum_{j=1}^{n} K_{nj}^1 \Delta U_j^1 - (K_n/k)\Delta T_2 - (M_n^1 + \Gamma) g \cos\phi^f \Delta\phi + \gamma_i^N \equiv \gamma_i^L + \gamma_i^N, \quad (41.1^*)$$

$$\gamma_2 = \Delta T_2 + g \sum_{j=1}^{m} M_j^2 [(\cos\beta^f X_j^2 + \sin\beta^f U_j^{2f})\Delta\beta - \cos\beta^f \Delta U_j^2] + \gamma_2^N \equiv \gamma_2^L + \gamma_2^N, \quad (41.2^*)$$

$$\gamma_3 = \Delta T_1 - \Delta T_2 + g\Gamma[(l^1 \cos\phi^f + U_n^{1f}\sin\phi^f)\Delta\phi - \cos\phi^f \Delta U_n^1] + g\sum_{i=1}^{n} M_i^1 [(\cos\phi^f X_i^1 + \sin\phi^f U_i^{1f})\Delta\phi - \cos\phi^f \Delta U_i^1] + \gamma_3^N \equiv \gamma_3^L + \gamma_3^N, \quad (41.3^*)$$

where the nonlinear parts, $p_i^N, q_i^N, \gamma_1^N, \gamma_2^N, \gamma_3^N$, can be easily obtained by examining eqns (39–41, 55–58, 61, 39*–41*).

From now on, the governing equation of the system, eqn (29), may be written symbolically as

$$\mathbf{A}^*\Delta\ddot{\omega} = \mathbf{F}\Delta\omega + \mathbf{G}\Delta\mathbf{T} + \mathbf{N}, \tag{62}$$

where $\mathbf{F}$ is a constant $(n+m+2)\times(n+m+2)$ matrix; $\mathbf{G}$ is a constant $(n+m+2)\times 2$ matrix; and $\mathbf{N}$ stands for the vector, of length $n+m+2$, of nonlinear functions. One may also rewrite eqn (62) as

$$\Delta\ddot{\omega} = \mathbf{A}(\mathbf{F}\Delta\omega + \mathbf{G}\Delta\mathbf{T} + \mathbf{N}). \tag{63}$$

It is noticed that so far no approximation whatsoever has been made and the order of the nonlinear functions, $\mathbf{N}$, is higher than or equal to two.

## 8. THE MEASUREMENTS

It is assumed that the position of the end effector, $(x^*, z^*)$, can be measured. The difference between the end effector and the target position can be obtained as

$$\begin{aligned}\delta_x &\equiv x^* - x^t \\ &= l^2(\sin\beta - \sin\beta^f) + l^1(\sin\phi - \sin\phi^f) \\ &\quad - \cos\beta U_m^2 + \cos\beta^f U_m^{2f} - \cos\phi U_n^1 \\ &\quad + \cos\phi^f U_n^{1f},\end{aligned} \tag{64}$$

$$\begin{aligned}\delta_z &\equiv z^* - z^t \\ &= l^2(\cos\beta - \cos\beta^f) + l^1(\cos\phi - \cos\phi^f) \\ &\quad + \sin\beta U_m^2 - \sin\beta^f U_m^{2f} + \sin\phi U_n^1 \\ &\quad - \sin\phi^f U_n^{1f}.\end{aligned} \tag{65}$$

It is seen that $\delta$ is a nonlinear function of the state variables. If a Taylor series expansion of $\delta$ is performed about the final position, the linear expressions of $\delta$ are obtained as

$$\begin{aligned}\delta_x &\approx l^2\cos\beta^f\Delta\beta + l^1\cos\phi^f\Delta\phi \\ &\quad + \sin\beta^f U_m^{2f}\Delta\beta + \sin\phi^f U_n^{1f}\Delta\phi \\ &\quad - \cos\beta^f\Delta U_m^2 - \cos\phi^f\Delta U_n^1,\end{aligned} \tag{66}$$

$$\begin{aligned}\delta_z &\approx -l^2\sin\beta^f\Delta\beta - l^1\sin\phi^f\Delta\phi \\ &\quad + \cos\beta^f U_m^{2f}\Delta\beta + \cos\phi^f U_n^{1f}\Delta\phi \\ &\quad + \sin\beta^f\Delta U_m^2 + \sin\phi^f\Delta U_n^1.\end{aligned} \tag{67}$$

Now, the governing equations and the measurements of the system in linear form can be symbolically written as

$$\Delta\ddot{\omega} = \mathbf{A}^f\mathbf{F}\Delta\omega + \mathbf{A}^f\mathbf{G}\Delta\mathbf{T}, \tag{63*}$$

$$\boldsymbol{\delta} = \mathbf{H}\Delta\omega, \tag{66*, 67*}$$

where

$$\mathbf{A}^f = \mathbf{A}[\omega^f], \tag{68}$$

and $\mathbf{H}$ is a constant $2\times(n+m+2)$ matrix. Based on the constant matrices, $\mathbf{A}^f\mathbf{F}$, $\mathbf{A}^f\mathbf{G}$, $\mathbf{H}$, a linear estimator can be constructed for the purpose of control.

## 9. DISCUSSION

It is noticed that, in the works of Lee *et al.* [7], Lee and Wang [8], for the control of a flexible robot arm, as well as in this work, the equations of the system and the measurements may be symbolically written as

$$\dot{\boldsymbol{\alpha}} = \mathbf{Y}(\boldsymbol{\alpha}, \mathbf{u}), \tag{69}$$

$$\boldsymbol{\delta} = \boldsymbol{\delta}(\boldsymbol{\alpha}), \tag{70}$$

where $\boldsymbol{\alpha}$ is a vector of state variables ($\boldsymbol{\alpha} = [\Delta\omega, \Delta\dot{\omega}]^T$ in this work); $\mathbf{u}$ is a vector of control variables ($\mathbf{u} = [\Delta T_1, \Delta T_2]^T$ in this work); $\boldsymbol{\delta}$ is the measurements—a nonlinear vector function of the state variables; $\mathbf{Y}$ is a nonlinear vector of function of the state and the control variables; $\mathbf{Y}$ and $\boldsymbol{\delta}$ may be written as

$$\dot{\boldsymbol{\alpha}} = \mathbf{P}\boldsymbol{\alpha} + \mathbf{R}\mathbf{u} + \mathbf{Y}^N(\boldsymbol{\alpha}, \mathbf{u}), \tag{71}$$

$$\boldsymbol{\delta} = \mathbf{H}\boldsymbol{\alpha} + \boldsymbol{\delta}^N(\boldsymbol{\alpha}). \tag{72}$$

In eqns (71–72), $\mathbf{P}$, $\mathbf{R}$, $\mathbf{H}$ are constant matrices; $\mathbf{Y}^N$ and $\boldsymbol{\delta}^N$ are nonlinear functions. Let $\hat{\boldsymbol{\alpha}}$ stand for the state variables of the estimator (observer) and let the observer and control law be expressed respectively as

$$\dot{\hat{\boldsymbol{\alpha}}} = \mathbf{P}\hat{\boldsymbol{\alpha}} + \mathbf{R}\mathbf{u} + \mathbf{L}(\boldsymbol{\delta} - \mathbf{H}\hat{\boldsymbol{\alpha}}), \tag{73}$$

$$\mathbf{u} = -\mathbf{J}\hat{\boldsymbol{\alpha}}, \tag{74}$$

where $\mathbf{L}$ is the estimate gain matrix and $\mathbf{J}$ is the control gain matrix. The gain matrices, $\mathbf{L}$ and $\mathbf{J}$, may be constructed based on $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{H}$ by using the pole-replacement method [13, 14], or by applying the optimal control theory [13, 15].

In this work, the detailed expressions of $\mathbf{P}$, $\mathbf{R}$, $\mathbf{H}$, $\mathbf{Y}^N$ and $\boldsymbol{\delta}^N$ have been derived rigorously for a two-link flexible robot arm moving in the vertical plane. If the system and the measurements had been linear, i.e., $\mathbf{Y}^N = \boldsymbol{\delta}^N = 0$, then any properly obtained gain matrices, $\mathbf{L}$ and $\mathbf{J}$, would have guaranteed the convergency and the stability of the solutions, in other words, the end effector eventually would have reached the target asymptotically. Because the system and the measurements are nonlinear, it is necessary to divide the processes of control into a first stage coarse control and a last stage fine control [7, 8]. A relatively simple coarse control law will bring the end effector to the neighborhood of the target and then the fine control law is activated to stabilize the whole system. Although finding the detailed expressions of the gain matrices is left for future study, we feel that there is no major difficulty in doing so by using the pole-placement method or by applying the optimal control theory.

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